Evolution of magnetic fields through cosmological perturbation theory

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Abstract. The origin of galactic and extra-galactic magnetic fields is an unsolved problem in modern cosmology. A possible scenario comes from the idea of these fields emerged from a small field, *a seed*, which was produced in the early universe (phase transitions, inflation, ...) and it evolves in time. Cosmological perturbation theory offers a natural way to study the evolution of primordial magnetic fields. The dynamics for this field in the cosmological context is described by a cosmic dynamo like equation, through the dynamo term. In this paper we get the perturbed Maxwell's equations and compute the energy momentum tensor to second order in perturbation theory in terms of gauge invariant quantities. Two possible scenarios are discussed, first we consider a FLRW background without magnetic field and we study the perturbation theory introducing the magnetic field as a perturbation. The second scenario, we consider a magnetized FLRW and build up the perturbation theory from this background. We compare the cosmological dynamo like equation in both scenarios.

Contents

1   Introduction

Magnetic fields have been observed on several scales in the universe. Galaxies and clusters of galaxies contain magnetic fields with strengths of $\sim 10^{-6}$ G [1, 2], fields within clusters are also likely to exist, with strengths of comparable amplitud [3, 4]. There is also, some evidence of their presence on scales of superclusters [5–8]. On the other hand, the possibility of cosmological magnetic field has been addressed comparing the CMB quadrupole with one induced by a constant magnetic field (in coherence scales of $\sim 1Mpc$), constraining the field amplitude to $B < 6.8 \times 10^{-9}(\Omega_m h^2)^{1/2}$ Gauss [9–11]. However, the origin of such large scale magnetic fields is still unknown. These fields are assumed to be increased and maintained by dynamo mechanism, but it needs a *seed* for the mechanism takes place [12]. Astrophysical mechanisms, as the Biermann battery have been used to explain how the magnetic field is mantained in objects as galaxies, stars and supernova remnants [13–16], but they are not likely correlated beyond galactic sizes [17]. It makes difficult to use only such mechanism to explain the origin of magnetic fields on cosmological scales. Possible explanations appear in differents context to generate the seed from the which the astrophysical mechanism start. For example, magnetic fields could be generated during primordial phase transitions (such as QCD, the electroweak or GUT), parity-violating processes which generates magnetic helicity or during inflation [18–27]. Magnetic fields also are generated during the radiation era in regions that have nonvanishing vorticity, this seed was proposed by Harrison [28–30]. Magnetic fields generation from density fluctuations in prerecombination era has been investigated in [31]. The advantage of these primordial processes is that they offer a wide range of coherence lengths (many of which are strongly constrained by Nucleosynthesis [32–34]), while the astrophysical mechanisms produce fields with characteristic scales of the size of astrophysical object. Recently has been found a lower limit of the large scale correlated magnetic field, this fact constrains models for the origin of cosmic magnetic fields, giving a possible evidence for their primordial origin [35–37].

Cosmological perturbation theory [38–40] is a powerful tool to understand the present properties of the large-scale structure of the Universe and their origin. It has been mainly used to predict effects on

the temperature distribution in the Cosmic Microwave Background (CMB) [41, 42] futhermore, linear perturbation theory combined with inflation suggest us that primordial fluctuations of the universe are adiabatic and Gaussian [43]. However, due to the high precision measurements reached in cosmology, higher order cosmological perturbation theory is required to test the current cosmological framework [44]. There are mainly two approaches to study higher order perturbative effects: one uses nonlinear theory and different manifestations of the separate universe approximation, using the ΔN formalism [45, 46], and the other one is the Bardeen's approach where metric and matter fields are expanded in a power series [47]. Within Bardeen's approach, a set of variables are determined in such way that have no gauge dependence. These are known in the literature as *gauge-invariant variables* which have been widely used in different cosmological scenarios [48, 49]. One important result of cosmological perturbation theory is the coupling between gravity and electromagnetic fields, finding a magneto-geometrical interaction that can change the evolution of the fields on large scales. An effect is the amplification of cosmic fields. Indeed, large scale magnetic fields in perturbed spatially open FLRW models decay as a^{-1}, a rate considerably slower than the standard a^{-2} [50–53], the hyperbolic geometry of these open FLRW models lead to the superadiabatic amplification on large scales [54]. The main goal in this paper is to study the late evolution of magnetic fields which were generated in early stages of the universe. We use the cosmological perturbation theory following the Bardeen's approach to find the perturbed Maxwell equations up to second order and also we get an equation like dynamo written in terms of gauge invariant variables to first and second order. Futhermore, we discussed the importance that the curvature and the gravitational potential play in the evolution of these fields. The paper is organized as follows: in the next section we briefly give an introduction of cosmological perturbation and we address the problem gauge in this theory. In section 3, we write the matter equations in the homogeneous and isotropic universe, which was used to generate the first and second order dynamical equations. In section 4, we define the first order gauge invariant variables for the perturbations not only in the matter (energy density, presure, magnetic and electric field) but also in the geometrical quantities (gravitational potential, curvature, shear ..). The first-order perturbation of the Maxwell's equations is reviewed in section 5 and together with the the Ohm's law allow us to find the cosmological dynamo equation to describe the evolution of the magnetic field. The derivation of second-order Maxwell's equations is given in section 7, and following the same methodology for the first-order case, we find the cosmological dynamo equation at second order written in terms of gauge invariant variables. In the section 8, we use an alternative approximation to the model considering a magnetic field in the FLRW background, we find that amplification effects of magnetic field appear at first order in the equations, besides of the absence of fractional orders. Also we discuss the differences found between both approaches. The final section 9 is devoted to discuss, the main results and the connection with future works.

2 The gauge problem in perturbation theory

Cosmological perturbation theory help us to find approximate solutions of the Einstein field equations through small desviations from an exact solution [55]. In this theory one works with two differents space-times, one is the real space-time $(\mathcal{M}, g_{\alpha\beta})$ which describes the perturbed universe and the other is the background space-time $(\mathcal{M}_0, g_{\alpha\beta}^{(0)})$ which is an idealization and is taken as reference to generate the real space-time. Then, the perturbation of any quantity Γ (e.g., energy density $\mu(x,t)$, 4-velocity $u^\alpha(x,t)$, magnetic field $B^i(x,t)$ or metric tensor $g_{\alpha\beta}$) is the difference between the value that the quantity Γ takes in the real space-time and the value in the background at a given point.[1] Now to determinate the perturbation in this quantity Γ, we must have a way to compare Γ (tensor on the real space-time) with $\Gamma^{(0)}$ (being $\Gamma^{(0)}$ the value on $\mathcal{M}_0$), this requires an assumption to identify points of $\mathcal{M}$ with those of $\mathcal{M}_0$. This is accomplished by assigning a mapping between these space-times called *gauge choice* given by a function $\mathcal{X} : \mathcal{M}_0(p) \longrightarrow \mathcal{M}(\bar{p})$ for any point $p \in \mathcal{M}_0$ and $\bar{p} \in \mathcal{M}$, which generate a pull-back

$$\mathcal{X}^* : \begin{array}{c} \mathcal{M} \\ T^*(\overline{p}) \end{array} \longrightarrow \begin{array}{c} \mathcal{M}_0 \\ T^*(p) \end{array}, \tag{2.1}$$

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[1]This difference should be taken in the same physical point.

thus, points on the real and background space-time can be compared through of $\mathcal{X}$. Then, the perturbation for $\varGamma$ is define as

$$\delta\varGamma(p) = \varGamma(\bar{p}) - \varGamma^{(0)}(p). \tag{2.2}$$

We see that the perturbation $\delta\varGamma$ is completely dependent of the gauge choice because the mapping determines the representation on $\mathcal{M}_0$ of $\varGamma(\bar{p})$. However, one can also choice another correspondence $\mathcal{Y}$ between these space-times so that $\mathcal{Y} : \mathcal{M}_0(q) \to \mathcal{M}(\overline{p})$, $(p \neq q)$.[2] A change of this identification map in the literature is called *gauge transformation*. The freedom to choose between different correspondences is due to the general covariance in General Relativity, which say that there is no preferred coordinate system in nature [56, 57]. Hence, this freedom will generate an arbitrariness in the value of $\delta\varGamma$ at any space-time point p, it is called *gauge problem* in the general relativistic perturbation theory and has been treated by [47, 58, 59]. This problem generates unphysical degree of freedom to the solutions in the theory, therefore one should fix the gauge or build up quantities that they do not dependent of this degree of freedom.

2.1 Gauge transformations and gauge invariant variables

To define the perturbation to a given order, we need to introduce the concept of Taylor expansion on a manifold and thus the metric and matter fields are expanded in a power series. Following [60–62], we consider a family of four-dimensional submanifolds $\mathcal{M}_\lambda$ with $\lambda \in \mathbb{R}$, embedded in a 5-dimensional manifold $\mathcal{N} = \mathcal{M} \times \mathbb{R}$. Each submanifold in the family represents a perturbed space-time and the background space-time is represented by the manifold $\mathcal{M}_0$ ($\lambda = 0$), and on these manifolds we consider that the Einstein field and Maxwell's equations are satisfied

$$\mathbb{E}[g_\lambda, T_\lambda] = 0 \quad \text{and} \quad M[F_\lambda, J_\lambda] = 0; \tag{2.3}$$

each tensor field $\varGamma_\lambda$ on a given manifold $\mathcal{M}_\lambda$ is extended to all manifold $\mathcal{N}$ through $\varGamma(p, \lambda) \equiv \varGamma_\lambda(p)$ to any $p \in \mathcal{M}_\lambda$ likewise the above equations are extended to $\mathcal{N}$.[3] We need to use a diffeomorphism such that the difference in the right side of eq. (2.2) can be done. We introduce an one-parameter group of diffeomorphisms $\mathcal{X}_\lambda$ which identifies points in the background with points in the real space-time labelled with the value λ. Each $\mathcal{X}_\lambda$ is a member of a flow $\mathcal{X}$ on $\mathcal{N}$ and it specifies a vector field X with the property $X^4 = 1$ everywhere (transverse to the $\mathcal{M}_\lambda$) besides other properties,[4] then points which lie on the same integral curve of X is to be regarded as the same point [59]. Therefore, according to the above, we get a definition for the tensor perturbation

$$\Delta\varGamma_\lambda \equiv \mathcal{X}_\lambda^*\varGamma|_{\mathcal{M}_0} - \varGamma_0. \tag{2.4}$$

At higher orders the Taylor expansion is given by [61],

$$\Delta^{\mathcal{X}}\varGamma_\lambda = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \delta_{\mathcal{X}}^{(k)} \varGamma - \varGamma_0 = \sum_{k=1}^{\infty} \frac{\lambda^k}{k!} \delta_{\mathcal{X}}^{(k)} \varGamma, \tag{2.5}$$

where

$$\delta_{\mathcal{X}}^{(k)}\varGamma = \left[\frac{d^k \mathcal{X}_\lambda^* \varGamma}{d\lambda^k} \right]_{\lambda=0, \mathcal{M}_0}. \tag{2.6}$$

Now, rewriting eq. (2.4) in the following form

$$\mathcal{X}_\lambda^*\varGamma|_{\mathcal{M}_0} = \varGamma_0 + \lambda\delta_{\mathcal{X}}^{(1)}\varGamma + \frac{\lambda^2}{2}\delta_{\mathcal{X}}^{(2)}\varGamma + \mathcal{O}(\lambda^3), \tag{2.7}$$

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[2]This is the active approach where transformations of the perturbed quantities are evaluated at the same coordinate point.

[3]In eq. (2.3), g_λ and T_λ are the metric and the matter fields on $\mathcal{M}_\lambda$, similarly F_λ and J_λ are the electromagnetic field and the four-current on $\mathcal{M}_\lambda$.

[4]Here we introduce a coordinate system x^α through a chart on $\mathcal{M}_\lambda$ with $\alpha = 0, 1, 2, 3$, thus, given a vector field on $\mathcal{N}$, it's the property that $X^4 = \lambda$ in this chart, while theirs other components remains arbitrary.

we see in the eqs. (2.6) and (2.7) that representation of Γ on $\mathcal{M}_0$ is splitting in the background value Γ_0 plus $\mathcal{O}(k)$ perturbations in the gauge $\mathcal{X}_\lambda$. Therefore, the k-th order $\mathcal{O}(k)$ in Γ depends on gauge $\mathcal{X}$. With this description the perturbations are fields living in the background. The first term in eq. (2.4) admits an expansion around of $\lambda = 0$ given by [61]

$$\mathcal{X}_\lambda^* \Gamma|_{\mathcal{M}_0} = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \mathcal{L}_X^k \Gamma|_{\mathcal{M}_0} = \exp(\lambda \mathcal{L}_X) \Gamma|_{\mathcal{M}_0}, \tag{2.8}$$

where $\mathcal{L}_X \Gamma$ is the Lie derivative of Γ with respect to a vector field X which generates the flow $\mathcal{X}$. If we define $\mathcal{X}_\lambda^* \Gamma|_{\mathcal{M}_0} \equiv \Gamma_\lambda^{\mathcal{X}}$ and doing the same for other gauge choice $\mathcal{Y}$, using eqs. (2.4)-(2.8), the tensor fields $\Gamma_\lambda^{\mathcal{X},\mathcal{Y}}$ can be written as

$$\Gamma_\lambda^{\mathcal{X}} = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \delta_{\mathcal{X}}^{(k)} \Gamma = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \mathcal{L}_X^k \Gamma|_{\mathcal{M}_0}, \tag{2.9}$$

$$\Gamma_\lambda^{\mathcal{Y}} = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \delta_{\mathcal{Y}}^{(k)} \Gamma = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \mathcal{L}_Y^k \Gamma|_{\mathcal{M}_0}, \tag{2.10}$$

if $\Gamma_\lambda^{\mathcal{X}} = \Gamma_\lambda^{\mathcal{Y}}$ for any arbitrary gauge $\mathcal{X}$ and $\mathcal{Y}$, we say that Γ is totally gauge invariant. We can also say that Γ is gauge invariant to order $n \geqslant 1$ iff satisfy $\delta_{\mathcal{Y}}^{(k)} \Gamma = \delta_{\mathcal{X}}^{(k)} \Gamma$, or in other way

$$\mathcal{L}_X \delta^{(k)} \Gamma = 0, \tag{2.11}$$

for any vector field X and $\forall k < n$. To first order $(k = 1)$ any scalar that is constant in the background or any tensor that vanished in the background are gauge invariant. This result is referred to as the *Stewart-Walker Lemma* [63], i.e., eq. (2.11) generalizes this Lemma. However, when Γ is not gauge invariant and we have two gauge choices $\mathcal{X}_\lambda$ and $\mathcal{Y}_\lambda$, the representation of $\Gamma|_{\mathcal{M}_0}$ is different depending of the gauge used. To transform the representation from a gauge choice $\mathcal{X}_\lambda^* \Gamma|_{\mathcal{M}_0}$ to another $\mathcal{Y}_\lambda^* \Gamma|_{\mathcal{M}_0}$ we use a map $\Phi_\lambda : \mathcal{M}_0 \to \mathcal{M}_0$ given by

$$\Phi_\lambda \equiv \mathcal{X}_{-\lambda} \circ \mathcal{Y}_\lambda \Rightarrow \Gamma_\lambda^{\mathcal{Y}} = \Phi_\lambda^* \Gamma_\lambda^{\mathcal{X}}, \tag{2.12}$$

thereby, the diffeomorphism Φ_λ induce a pull-back Φ_λ^* which changes the representation $\Gamma_\lambda^{\mathcal{X}}$ of Γ in a gauge $\mathcal{X}_\lambda$ to the representation $\Gamma_\lambda^{\mathcal{Y}}$ of Γ in a gauge $\mathcal{Y}_\lambda$. Now, following [64] and using the Baker-Campbell-Haussdorf formula [65], we can generalize eq. (2.8) to write $\Phi_\lambda^* \Gamma_\lambda^{\mathcal{X}}$ in the following way

$$\Phi_\lambda^* \Gamma_\lambda^{\mathcal{X}} = \exp\left(\sum_{k=1}^{\infty} \frac{\lambda^k}{k!} \mathcal{L}_{\xi_k}\right) \Gamma_\lambda^{\mathcal{X}}, \tag{2.13}$$

where ξ_k is any vector field on $\mathcal{M}_\lambda$. Substitute eq.(2.13) in eq.(2.12), we have explicitly that

$$\Gamma_\lambda^{\mathcal{Y}} = \Gamma_\lambda^{\mathcal{X}} + \lambda \mathcal{L}_{\xi_1} \Gamma_\lambda^{\mathcal{X}} + \frac{\lambda^2}{2} \left(\mathcal{L}_{\xi_1}^2 + \mathcal{L}_{\xi_2}\right) \Gamma_\lambda^{\mathcal{X}} + \mathcal{O}(\lambda^3). \tag{2.14}$$

Thus, if substitute eq.(2.9) and eq.(2.10) into eq.(2.14), the relation to first and second order perturbations of Γ in two different gauge choice are given by

$$\delta_{\mathcal{Y}}^{(1)} \Gamma - \delta_{\mathcal{X}}^{(1)} \Gamma = \mathcal{L}_{\xi_1} \Gamma_0, \tag{2.15}$$

$$\delta_{\mathcal{Y}}^{(2)} \Gamma - \delta_{\mathcal{X}}^{(2)} \Gamma = 2\mathcal{L}_{\xi_1} \delta_{\mathcal{X}}^{(1)} \Gamma + \left(\mathcal{L}_{\xi_1}^2 + \mathcal{L}_{\xi_2}\right) \Gamma_0, \tag{2.16}$$

where the generators of the gauge transformation Φ are

$$\xi_1 = Y - X \quad \text{and} \quad \xi_2 = [X, Y]. \tag{2.17}$$

The gauge transformation given by the eqs. (2.15) and (2.16) are quite general, to first order we see that Γ is gauge invariant iff $\mathcal{L}_{\xi_1} \Gamma_0 = 0$, while to second order one must have another condition $\mathcal{L}_{\xi_1} \delta_{\mathcal{X}}^{(1)} \Gamma = 0$, and so at high orders. We will apply the formalism described above to the Robertson-Walker metric, where k does mention to the expansion order.

3 FLRW background

We assume that at zero order (background), the universe is well described by a spatially flat Friedman-Lemaître-Robertson-Walker metric (FLRW)

$$ds^2 = a^2(\tau)\left(-d\tau^2 + \delta_{ij}dx^i dx^j\right), \tag{3.1}$$

with $a(\tau)$ the scale factor. Hereafter the Greek indices run from 0 to 3, and the Latin ones run from 1 to 3, we will work with conformal time τ, and a prime denotes the derivative with respect to τ. The Einstein's components tensor in this background are given by

$$G_0^0 = -\frac{3H^2}{a^2}, \tag{3.2}$$

$$G_j^i = -\frac{1}{a^2}\left(2\frac{a''}{a} - H^2\right)\delta_j^i, \tag{3.3}$$

$$G_i^0 = G_0^i = 0, \tag{3.4}$$

with $H = \frac{a'}{a}$ the Hubble parameter. We consider the background filled with a single barotropic fluid where the energy momentum tensor is

$$T_{(fl)\,\nu}^{\mu} = \left(\mu_{(0)} + P_{(0)}\right)u_{(0)}^{\mu}u_{\nu}^{(0)} + P_{(0)}\delta_{\nu}^{\mu}, \tag{3.5}$$

with $\mu_{(0)}$ the energy density and $P_{(0)}$ the pressure. The comoving observers are defined by the four-velocity $u^{\nu} = (a^{-1}, 0, 0, 0)$ with $u^{\nu}u_{\nu} = -1$ and the conservation law for the fluid is

$$\mu'_{(0)} + 3H(\mu_{(0)} + P_{(0)}) = 0. \tag{3.6}$$

To deal with the magnetic field, the space-time under study is the fluid permeated by a weak magnetic field,[5] it is a stochastic field and can be treat as a perturbation on the background [66, 67]. Due to the magnetic field has no background contribution, the electromagnetic energy momentum tensor is automatically gauge invariant (see eq.(2.15)). Now we use the spatial part of Ohm's law [68], which is the current proyected in the slices

$$(g_{\mu i} + u_{\mu}u_i)\,j^{\mu} = \sigma g_{\lambda i}g_{\alpha\mu}F^{\lambda\alpha}u^{\mu}, \tag{3.7}$$

where $j^{\mu} = (\varrho, J^i)$ is the 4-current and $F^{\lambda\alpha}$ is the electromagnetic tensor given by

$$F^{\lambda\alpha} = \frac{1}{a^2(\tau)}\begin{pmatrix} 0 & E^i & E^j & E^k \\ -E^i & 0 & B^k & -B^j \\ -E^j & -B^k & 0 & B^i \\ -E^k & B^j & -B^i & 0 \end{pmatrix}. \tag{3.8}$$

At zero orden in eq. (3.7) we find the usual Ohm's law which gives us the relation between the 3-current and the electric field

$$J_i = \sigma E_i, \tag{3.9}$$

where σ is the conductivity. We work under MHD approximation, thus in large scales the plasma is globally neutral and charge density is neglected ($\varrho = 0$) [3]. If the conductivity is infinite ($\sigma \to \infty$) in the early universe [69, 70], then eq. (3.7) say us that electric field must vanish ($E_i = 0$) in order to keep the current density finite [71, 72]. Hovewer, the current also should be zero ($J_i = 0$) because a nonzero current involves a movement of charge particles which breaks down the isotropy in the background.

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[5]With the property $B_{(0)}^2 \ll \mu_{(0)}$ [66].

4 Gauge invariant variables at first order

We write down the perturbations on a spatially flat Robertson-Walker background. We shall consider the fluctuations of the metric and matter fields. The perturbative expansion at $k-$th order of the matter quantities are given by

$$\mu = \mu_{(0)} + \sum_{k=1}^{\infty} \frac{1}{k!} \mu_{(k)}, \tag{4.1}$$

$$B^2 = \sum_{k=1}^{\infty} \frac{1}{k!} B_{(k)}^2, \tag{4.2}$$

$$E^2 = \sum_{k=1}^{\infty} \frac{1}{k!} E_{(k)}^2, \tag{4.3}$$

$$P = P_{(0)} + \sum_{k=1}^{\infty} \frac{1}{k!} P_{(k)}, \tag{4.4}$$

$$B^i = \frac{1}{a(\tau)^2} \left(\sum_{k=1}^{\infty} \frac{1}{k!} B_{(k)}^i \right), \tag{4.5}$$

$$E^i = \frac{1}{a(\tau)^2} \left(\sum_{k=1}^{\infty} \frac{1}{k!} E_{(k)}^i \right), \tag{4.6}$$

$$u^\mu = \frac{1}{a(\tau)} \left(\delta_0^\mu + \sum_{k=1}^{\infty} \frac{1}{k!} v_{(k)}^\mu \right), \tag{4.7}$$

$$j^\mu = \frac{1}{a(\tau)} \left(\sum_{k=1}^{\infty} \frac{1}{k!} j_{(k)}^\mu \right), \tag{4.8}$$

Where the fields used in above formulae are the average ones (i.e. $B^2 = \langle B^2 \rangle$).[6] We also consider the perturbations about a FLRW background, so that the metric tensor is given by

$$g_{00} = -a(\tau)^2 \left(1 + 2 \sum_{k=1}^{\infty} \frac{1}{k!} \psi^{(k)} \right), \tag{4.9}$$

$$g_{oi} = a(\tau)^2 \sum_{k=1}^{\infty} \frac{1}{k!} \omega_i^{(k)}, \tag{4.10}$$

$$g_{ij} = a(\tau)^2 \left[\left(1 - 2 \sum_{k=1}^{\infty} \frac{1}{k!} \phi^{(k)} \right) \delta_{ij} + \sum_{k=1}^{\infty} \frac{1}{k!} \chi_{ij}^{(k)} \right]. \tag{4.11}$$

The perturbations are splitting into scalar, transverse vector part, and transverse trace-free tensor

$$\omega_i^{(k)} = \partial_i \omega^{(k)\|} + \omega_i^{(k)\perp}, \tag{4.12}$$

with $\partial^i \omega_i^{(k)\perp} = 0$. Similarly we can split $\chi_{ij}^{(k)}$ as

$$\chi_{ij}^{(k)} = D_{ij} \chi^{(k)\|} + \partial_i \chi_j^{(k)\perp} + \partial_j \chi_i^{(k)\perp} + \chi_{ij}^{(k)\top}, \tag{4.13}$$

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[6]This happens because the average evolves exactly like B^2 [73].

for any tensor quantity.[7] Following [74], one can find the scalar *gauge invariant variables* at first order given by

$$\Psi^{(1)} \equiv \psi^{(1)} + \frac{1}{a}\left(\mathcal{S}^{||}_{(1)}a\right)', \tag{4.14}$$

$$\Phi^{(1)} \equiv \phi^{(1)} + \frac{1}{6}\nabla^2\chi^{(1)} - H\mathcal{S}^{||}_{(1)}, \tag{4.15}$$

$$\Delta^{(1)} \equiv \mu_{(1)} + \left(\mu_{(0)}\right)'\mathcal{S}^{||}_{(1)}, \tag{4.16}$$

$$\Delta^{(1)}_P \equiv P_{(1)} + \left(P_{(0)}\right)'\mathcal{S}^{||}_{(1)}, \tag{4.17}$$

with $\mathcal{S}^{||}_{(1)} \equiv \left(\omega^{||(1)} - \frac{(\chi^{||(1)})'}{2}\right)$. The vector modes are

$$v^i_{(1)} \equiv v^i_{(1)} + \left(\chi^i_{\perp(1)}\right)', \tag{4.18}$$

$$\vartheta^{(1)}_i \equiv \omega^{(1)}_i - \left(\chi^{\perp(1)}_i\right)', \tag{4.19}$$

$$\mathcal{V}^i_{(1)} \equiv \omega^i_{(1)} + v^i_{(1)}. \tag{4.20}$$

Another gauge invariant variables are the 3-current, the charge density and the electric and magnetic fields, because they vanish in the background. The tensor quantities are also gauge invariant because they are null in the background (see eq.(2.15)).

4.1 The Ohm's law and the energy momentum tensor

Using eq. (3.7) the Ohm's law at first order is

$$J^{(1)}_i = \sigma E^{(1)}_i. \tag{4.21}$$

At first order we suppose the conductivity of the medium finite (real MHD) therefore the electric field and the 3-current are nonzero. Now the electromagnetic energy momentum tensor is

$$T^0_{(em)\,0} = -\frac{1}{8\pi}\left(B^2_{(1)} + E^2_{(1)}\right), \tag{4.22}$$

$$T^i_{(em)0} = 0, \quad T^0_{(em)i} = 0, \tag{4.23}$$

$$T^i_{(em)\,l} = \frac{1}{4\pi}\left[\frac{1}{6}\left(B^2_{(1)} + E^2_{(1)}\right)\delta^i_l + \Pi^{i(1)}_{l(em)}\right], \tag{4.24}$$

where $\Pi^{i(1)}_{l(em)} = \frac{1}{3}\left(B^2 + E^2\right)\delta^i_l - B_lB^i - E_lE^i$ is the anisotropic stresses which is gauge invariant by definition eq.(2.15), this term is important to constrains the total magnetic energy because it's source of gravitational waves [33]. We see that electromagnetic energy density appear like a cuadratic term in the energy momentum tensor, this means that electromagnetic field should be regarded a $\frac{1}{2}$ order perturbation.[8] Using eq. (3.5) and considering the fluctuations of the matter fields eqs. (4.1) and (4.4), the energy momentum tensor for the fluid is given by

$$T^0_{(fl)\,0} = -\Delta^{(1)} + \left(\mu_{(0)}\right)'\mathcal{S}^{||}_{(1)}, \tag{4.25}$$

$$T^i_{(fl)\,0} = \left(\mu_0 + P_0\right)\left(\mathcal{V}^i_{(1)} - \vartheta^i_{(1)} - \left(\chi^i_{\perp(1)}\right)'\right), \tag{4.26}$$

$$T^0_{(fl)\,i} = -\left(\mu_0 + P_0\right)\mathcal{V}^{(1)}_i, \tag{4.27}$$

$$T^i_{(fl)\,j} = \left(\Delta^{(1)}_P - \left(P_{(0)}\right)'\mathcal{S}^{||}_{(1)}\right)\delta^i_j + \Pi^{i(1)}_{j(fl)}, \tag{4.28}$$

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[7]With $\partial^i\chi^{(k)\top}_{ij} = 0$, $\chi^{(k)i}_i = 0$ and $D_{ij} \equiv \partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2$.

[8]Therefore the magnetic field should be splitting as $B^i = \frac{1}{a(\tau)^2}\left(B^i_{(\frac{1}{2})} + B^i_{(1)} + B^i_{(\frac{3}{2})} + ...\right)$, see [75, 76].

where $\Pi^{i(1)}_{j(f)}$ is the anisotropic stress tensor [77]. The above equations are written in terms of gauge invariant variables plus terms as $\mathcal{S}^{||}_{(1)}$ that depend of the gauge choice.

4.2 The conservation equations

The total energy momentum conservation equation $\mathcal{T}^{\alpha}_{\beta;\alpha} = 0$ can be split in each component which is not conserved independently

$$\mathcal{T}^{\alpha}_{\beta;\alpha} = T^{\alpha(f)}_{\beta;\alpha} + T^{\alpha(E.M)}_{\beta;\alpha} = 0, \tag{4.29}$$

where

$$T^{\alpha(E.M)}_{\beta;\alpha} = F_{\beta\alpha}j^{\alpha}. \tag{4.30}$$

Using the eqs. (4.25) and (4.28), the continuity equation $\mathcal{T}^{\alpha}_{0;\alpha} = 0$ is given by

$$\left(\Delta^{(1)}\right)' + 3H\left(\Delta^{(1)}_P + \Delta^{(1)}\right) - 3\left(\Phi^{(1)}\right)'(P_0 + \mu_0) + (P_0 + \mu_0)\nabla^2 \upsilon^{(1)} - 3H\left(P_{(0)} + \mu_{(0)}\right)'\mathcal{S}^{||}_{(1)}$$
$$- \left(\left(\mu_{(0)}\right)'\mathcal{S}^{||}_{(1)}\right)' + (P_0 + \mu_0)\left(-\frac{1}{2}\nabla^2\chi^{(1)} + 3H\mathcal{S}^{||}_{(1)}\right)' - (P_0 + \mu_0)\nabla^2\left(\frac{1}{2}\chi^{||(1)}\right)' = 0. \tag{4.31}$$

The Navier-Stokes equation $\mathcal{T}^{\alpha}_{i;\alpha} = 0$ is

$$\left(\mathcal{V}^{(1)}_i\right)' + \frac{\left(\mu_{(0)} + P_{(0)}\right)'}{\left(\mu_{(0)} + P_{(0)}\right)}\mathcal{V}^{(1)}_i - 4H\mathcal{V}^{(1)}_i + \partial_i\Psi^{(1)} - \frac{\partial_i\left(\Delta^{(1)}_P - \left(P_{(0)}\right)'\mathcal{S}^{||}_{(1)}\right) + \partial_l\Pi^{(1)l}_{(fl)i}}{\left(\mu_{(0)} + P_{(0)}\right)} - \partial_i\frac{1}{a}\left(\mathcal{S}^{||}_{(1)}a\right)' = 0. \tag{4.32}$$

The above eqs. (4.31) and (4.32) are written is terms of gauge invariant variables in according to [68, 77, 78], we see that doesn't exist contribution of electromagnetic terms to the conservation equations. In [29, 79] the energy-momentum tensor of each component is not conserved independently and its divergence has a source term that takes into account the energy and momentum transfer among the components of the photon, electron, proton and electromagnetic field $T^{\alpha(f)}_{\beta;\alpha} = Q_\beta$.

5 Maxwell's equations and the cosmological dynamo equation

The Maxwell's equations are written as

$$\nabla_\alpha F^{\alpha\beta} = j^\beta, \quad \nabla_{[\gamma}F_{\alpha\beta]} = 0. \tag{5.1}$$

Using eq. (5.1) and the pertubation equations for the metric and electromagnetic fields, we find the non-homogeneous Maxwell's equations

$$\partial_i E^i_{(1)} = a\varrho_{(1)}, \tag{5.2}$$

$$\epsilon^{ilk}\partial_l B^k_{(1)} = \left(E^i_{(1)}\right)' + 2HE^i_{(1)} + aJ^i_{(1)}, \tag{5.3}$$

and the homogeneous Maxwell's equation

$$B'_{k(1)} + 2HB^{(1)}_k + \epsilon^{ij}_k\partial_i E^{(1)}_j = 0, \tag{5.4}$$

$$\partial^i B^{(1)}_i + \partial^j B^{(1)}_j + \partial^k B^{(1)}_k = 0, \tag{5.5}$$

which is in according to [80]. Now using the above equations eqs. (5.2),(5.4),(5.3) and (5.5) with the ohm's law eq. (4.21), we get an equation which gives us the evolution of magnetic field at first order, it's called *dynamo equation*:

$$\left(B^{(1)}_k\right)' + 2HB^{(1)}_k + \eta\left[\nabla\times\left(\nabla\times B^{(1)} - \left(E^{(1)}\right)' - 2HE^{(1)}\right)\right]_k = 0, \tag{5.6}$$

with $\eta = \frac{1}{4\pi\sigma}$ the diffusion coefficient [4]. The eq. (5.6) is similar to dynamo equation in MHD but it's in the cosmological context [3, 4]. This equation has one term that depends on η *diffusion term* which takes into account the dissipation phenomena of the magnetic field (the electric field in this term in general is dropped if we neglect the displacement current). We should have into account that η is a expansion parameter (due to $\frac{1}{\sigma}$ is small), from eq. (5.6) we see that for a finite η the diffusion term should not be neglected. Is important to remark one should take care with the assumption $\eta = 0$ to any scale, because it could break to some scale [70].

6 Generalization at second order

Following to [62] is introduced the variable $\delta^{(2)}\boldsymbol{T}$ defined by

$$\delta_{\mathcal{X}}^{(2)}\boldsymbol{T} \equiv \delta_{\mathcal{X}}^{(2)}\Gamma - 2L_X\left(\delta_{\mathcal{X}}^{(1)}\Gamma\right) + L_X^2\Gamma_0; \tag{6.1}$$

inspecting the gauge transformation eq. (2.16) one can see that $\delta^{(2)}\boldsymbol{T}$ is transformed as

$$\delta_{\mathcal{Y}}^{(2)}\boldsymbol{T} - \delta_{\mathcal{X}}^{(2)}\boldsymbol{T} = L_\sigma\Gamma_0, \tag{6.2}$$

with $\sigma = \xi_2 + [\xi_1, X]$ and X is the gauge dependence part in linear order perturbation. The gauge transformation rule eq. (6.2), is identical to the gauge transformation at linear order eq. (2.15). This property is general not only happens in the case of cosmological perturbation around FLRW metric, it's the key to extend this theory to second order

$$L\left[\delta^2\boldsymbol{T}\right] = S\left[\delta\boldsymbol{T}, \delta\boldsymbol{T}\right].$$

We shall see that first and second order equations are similar however, the last have as sources couples between linear perturbations variables. Using eqs. (2.16) and (6.2) we can arrive to the gauge invariant quantities at second order. The scalar gauge invariant at second order are given by

$$\Psi^{(2)} \equiv \psi^{(2)} + \frac{1}{a}\left(\mathcal{S}_{(2)}^{||}a\right)' + \mathcal{T}^1(\mathcal{O}^{(2)}), \tag{6.3}$$

$$\Phi^{(2)} \equiv \phi^{(2)} + \frac{1}{6}\nabla^2\chi^{(2)} - H\mathcal{S}_{(2)}^{||} + \mathcal{T}^2(\mathcal{O}^{(2)}), \tag{6.4}$$

$$\Delta_\mu^{(2)} \equiv \mu_{(2)} + \left(\mu_{(0)}\right)'\mathcal{S}_{(2)}^{||} + \mathcal{T}^3(\mathcal{O}^{(2)}), \tag{6.5}$$

$$\Delta_\varrho^{(2)} \equiv \varrho^{(2)} + \mathcal{T}^4(\mathcal{O}^{(2)}), \tag{6.6}$$

$$\Delta_B^{(2)} \equiv B_{(2)}^2 + \mathcal{T}^4(\mathcal{O}^{(2)}), \tag{6.7}$$

$$\Delta_E^{(2)} \equiv E_{(2)}^2 + \mathcal{T}^6(\mathcal{O}^{(2)}), \tag{6.8}$$

$$\upsilon^{(2)} \equiv v^{(2)} + \left(\frac{1}{2}\chi^{||(2)}\right)' + \mathcal{T}^7(\mathcal{O}^{(2)}), \tag{6.9}$$

with $\mathcal{S}_{(2)}^{||} \equiv \left(\omega^{||(2)} - \frac{(\chi^{||(2)})'}{2}\right) + \mathcal{T}^8(\mathcal{O}^{(2)})$. Vector modes found are as follows

$$\upsilon_{(2)}^i \equiv v_{(2)}^i + \left(\chi_{\perp(2)}^i\right)' + \mathcal{T}^9(\mathcal{O}^{(2)}), \tag{6.10}$$

$$\vartheta_i^{(2)} \equiv \omega_i^{(2)} - \left(\chi_i^{\perp(2)}\right)' + \mathcal{T}^{10}(\mathcal{O}^{(2)}), \tag{6.11}$$

$$\mathcal{V}_{(2)}^i \equiv \omega_{(2)}^i + v_{(2)}^i + \mathcal{T}^{11}(\mathcal{O}^{(2)}), \tag{6.12}$$

$$\mathcal{B}_{(2)}^i \equiv B_{(2)}^i + \frac{2}{a^2}B_{(1)}^j\partial_j\chi_{(1)\perp}^i - \frac{2}{a^2}\partial_l B_{(1)}^i\chi_{(1)\perp}^l, \tag{6.13}$$

$$\mathcal{E}_{(2)}^i \equiv E_{(2)}^i + \frac{2}{a^2}E_{(1)}^j\partial_j\chi_{(1)\perp}^i - \frac{2}{a^2}\partial_l E_{(1)}^i\chi_{(1)\perp}^l, \tag{6.14}$$

$$\mathcal{J}_{(2)}^i \equiv J_{(2)}^i + \mathcal{T}^{12}(\mathcal{O}^{(2)}), \tag{6.15}$$

$$\Pi_{ij}^{(2)T} \equiv \Pi_{ij}^{(2)fl} + \Pi_{ij}^{(2)em} + \mathcal{T}^{13}(\mathcal{O}^{(2)}), \tag{6.16}$$

where we calculate the gauge invariant quantities eqs. (6.13) and (6.14) for electromagnetic fields because they are gauge dependent at second order. In the eqs. (6.3) to (6.16) we have written the gauge invariant quatities at second order. We can see that these variables are similar to these ones at first orden but in this case appear sources as $T^k(\mathcal{O}^{(2)})$ that depend of the gauge choice and the couples of terms at first order. The explict calcules of $\mathcal{T}^k(\mathcal{O}^{(2)})$ can be seen in [61, 74].

6.1 The Ohm's law and the energy momentum tensor

Using eqs. (3.7), (4.5) and (4.6), we get the Ohm's law at second order

$$
\begin{aligned}
\mathcal{J}_i^{(2)} &- 4J_i^{(1)}\Phi^{(1)} - \varrho^{(1)}v_i^{(1)} + \boldsymbol{S}_i^1(\mathcal{O}^{(2)}) \\
&= 2\sigma\left(\left(\mathcal{V}_{(1)} \times B^{(1)}\right)_i - 2E_i^{(1)}\left(\Phi^{(1)} - \frac{1}{2}\Psi^{(1)}\right) + \frac{1}{2}\mathcal{E}_i^{(2)} + \boldsymbol{S}_i^2(\mathcal{O}^{(2)})\right).
\end{aligned}
\tag{6.17}
$$

In this case we see that 3-current has a type Lorentz term and shows couples between first order terms that can affect the evolution of the current. Hereafter the functions $\boldsymbol{S}_i^n(\mathcal{O}^{(2)})$ with $n \in \mathbb{Z}$ gives us the gauge dependence. Doing the expansion at second order in the fluid energy momentum tensor, one can find the following expression

$$
T_{(2)\,0}^0 = -\frac{\Delta_\mu^{(2)}}{2} - (\mu_0 + P_0)\left(v_l^{(1)}v_{(1)}^l + \vartheta_l^{(1)}v_{(1)}^l\right) + \boldsymbol{S}^3(\mathcal{O}^{(2)}),
\tag{6.18}
$$

$$
T_{(2)\,0}^i = -(\mu_0 + P_0)\left(\frac{\mathcal{V}_{(2)}^i - \vartheta_{(2)}^i}{2} + \Psi^{(1)}v_{(1)}^i\right) - \left(\Delta_\mu^{(1)} + \Delta_P^{(1)}\right)v_{(1)}^i + \boldsymbol{S}_4^i\left(\mathcal{O}^{(2)}\right),
\tag{6.19}
$$

$$
\begin{aligned}
T_{(2)\,i}^0 = &-(\mu_0 + P_0)\left(\frac{\mathcal{V}_i^{(2)}}{2} - 2\vartheta_i^{(2)}\Psi^{(1)} - 2v_i^{(1)}\Phi^{(1)} + v_{(1)}^j\chi_{ij}^{(1)} - v_i^{(1)}\Psi^{(1)}\right) \\
&- \left(\Delta_\mu^{(1)} + \Delta_P^{(1)}\right)\mathcal{V}_i^{(1)} + \boldsymbol{S}_i^5\left(\mathcal{O}^{(2)}\right),
\end{aligned}
\tag{6.20}
$$

$$
T_{(2)\,j}^i = \frac{1}{2}\Delta_P^{(2)}\delta_j^i + \frac{1}{2}\Pi_j^{i(2)*} + (\mu_0 + P_0)\left(v_j^{(1)}v_{(1)}^i + \vartheta_j^{(1)}v_{(1)}^i\right) + \boldsymbol{S}_j^{6i}\left(\mathcal{O}^{(2)}\right),
\tag{6.21}
$$

which is in according to [78]. Now consider eq. (5.4) the electromagnetic momentum tensor at second order is

$$
T_{(em)\,0}^0 = -\frac{1}{8\pi}\left(\Delta_E^{(2)} + \Delta_B^{(2)} + \boldsymbol{S}^8(\mathcal{O}^{(2)})\right),
\tag{6.22}
$$

$$
T_{(em)0}^i = \frac{1}{4\pi}\left[-\epsilon^{ikm}E_k^{(1)}B_{(1)}^m + \boldsymbol{S}_9^i(\mathcal{O}^{(2)})\right],
\tag{6.23}
$$

$$
T_{(em)i}^0 = \frac{1}{4\pi}\left[\epsilon_i^{km}E_k^{(1)}B_{(1)}^m + \boldsymbol{S}_{10i}(\mathcal{O}^{(2)})\right],
\tag{6.24}
$$

$$
T_{(em)\,l}^i = \frac{1}{4\pi}\left[\frac{1}{6}\left(\Delta_E^{(2)} + \Delta_B^{(2)} + \boldsymbol{S}_{4l}^i(\mathcal{O}^{(1)})\right)\delta_l^i + \Pi_{l(em)}^{i(2)*} + \boldsymbol{S}_{11l}^i(\mathcal{O}^{(2)})\right].
\tag{6.25}
$$

Using eq. (4.29) the continuity equation is given by

$$
\begin{aligned}
\left(\Delta_\mu^{(2)}\right)' &+ 3H\left(\Delta_P^{(2)} + \Delta_\mu^{(2)}\right) - 3\left(\Phi^{(2)}\right)'(P_0 + \mu_0) \\
&+ (P_0 + \mu_0)\nabla^2 v^{(2)} - \boldsymbol{S}_{12}\left(\mathcal{O}^{(2)}\right) = -a^4\left(2E_i^{(1)}J_{(1)}^i\right),
\end{aligned}
\tag{6.26}
$$

and the Navier-stokes equation

$$
\begin{aligned}
\left(\mathcal{V}_i^{(2)}\right)' &+ \frac{\left(\mu_{(0)} + P_{(0)}\right)'}{\left(\mu_{(0)} + P_{(0)}\right)}\mathcal{V}_i^{(2)} - 4H\mathcal{V}_i^{(2)} + \partial_i\Psi^{(2)} - \frac{\partial_i\Delta_P^{(2)} + \partial_l\Pi_{i(fl)}^{(2)l}}{\left(\mu_{(0)} + P_{(0)}\right)} \\
&+ \boldsymbol{S}_{13}\left(\mathcal{O}^{(2)}\right) = \frac{a^4}{\left(\mu_{(0)} + P_{(0)}\right)}\left(2E_i^{(1)}\varrho_{(1)} + 2\epsilon_{ijk}J_{(1)}^j B_k^{(1)}\right);
\end{aligned}
\tag{6.27}
$$

which is in according to [78, 81]. In this case we see that electromagnetic fields affect the evolution of matter energy density $\Delta_{\mu}^{(2)}$ and the peculiar velocity $\mathcal{V}_i^{(2)}$ therefore, these fields influence the large structure formation and can leave imprints on the temperature anisotropy pattern of the CMB [68, 82–84].

7 The Maxwell's equations and the cosmological dynamo at second order

Using the eq. (4.21), the non homogeneous Maxwell's equations are

$$\partial_i \mathcal{E}_{(2)}^i = -4E_{(1)}^i \partial_i \left(\Psi^{(1)} - 3\Phi^{(1)} \right) - \boldsymbol{S}_{14} \left(\mathcal{O}^{(2)} \right) + a\Delta_{\varrho}^{(2)}, \tag{7.1}$$

$$\left(\nabla \times \mathcal{B}^{(2)} \right)^i = \left(\mathcal{E}_{(2)}^i \right)' + 2H\mathcal{E}_{(2)}^i + 2E_{(1)}^i \left(2 \left(\Psi^{(1)} \right)' - 6 \left(\Phi^{(1)} \right)' \right)$$
$$+ 2 \left(2\Psi^{(1)} - 6\Phi^{(1)} \right) \left(\nabla \times B_{(1)} \right)^i + a\mathcal{J}_{(2)}^i + \boldsymbol{S}_{15}^i \left(\mathcal{O}^{(2)} \right). \tag{7.2}$$

While the homogeneous Maxwell's equations

$$\partial_i \mathcal{B}_i^{(2)} + \partial_j \mathcal{B}_j^{(2)} + \partial_k \mathcal{B}_k^{(2)} = -\boldsymbol{S}^{16} \left(\mathcal{O}^{(2)} \right), \tag{7.3}$$

$$\left(\mathcal{B}_k^{(2)} \right)' + 2H \left(\mathcal{B}_k^{(2)} \right) + \left(\nabla \times \mathcal{E}_{j(2)} \right)_k = -\boldsymbol{S}_k^{17} \left(\mathcal{O}^{(2)} \right); \tag{7.4}$$

Where the $\boldsymbol{S}_k^n$ terms carry out the gauge dependence. Using the above Maxwell's equations together with the Ohm's law at second order and following the same methodology for the first order case, we get the cosmological dynamo equation which describes the evolution of the magnetic field at second order

$$\left(\mathcal{B}_k^{(2)} \right)' + 2H \left(\mathcal{B}_k^{(2)} \right) + \eta \Bigg[\nabla \times \left(\frac{1}{a} \left(\left(\nabla \times \mathcal{B}^{(2)} \right) - 2E_{(1)} \left(2 \left(\Psi^{(1)} \right)' - 6 \left(\Phi^{(1)} \right)' \right) \right. \right.$$
$$- \left(\mathcal{E}_{(2)} \right)' - 2H\mathcal{E}_{(2)} - 2 \left(\nabla \times B_{(1)} \left(2\Psi^{(1)} - 6\Phi^{(1)} \right) \right) - \boldsymbol{S}_{15} \left(\mathcal{O}^{(1)} \right) \right) - \varrho^{(1)} \upsilon^{(1)} + \boldsymbol{S}^1 \left(\mathcal{O}^{(2)} \right) \Bigg) \Bigg]_k$$
$$+ \left(\nabla \times \left[-2 \left(\mathcal{V}_{(1)} \times B^{(1)} \right) - 2E^{(1)}\Psi^{(1)} - 2\boldsymbol{S}^2 \left(\mathcal{O}^{(1)} \right) \right] \right)_k = -\boldsymbol{S}_k^{17} \left(\mathcal{O}^{(2)} \right). \tag{7.5}$$

We find that perturbations in the space-time play an important role in the evolution of primordial magnetic fields. In this case we see from eqs. (5.6) and (7.5) dependent on geometrical quantities (perturbation in the gravitational potential, curvature, velocity ...). We know that these quantities evolve according to the Einstein field equations (the Einstein field equation to second order are given in [59, 81]). In this way, Eq. (7.5) tell us how the magnetic field evolve according to scale of the perturbation. In subhorizon scale, the contrast density and the geometrical quantities grow, hence, the dynamo term should amplify the magnetic field. As a final comment we point out that in order to solve the dynamo like equation for the magnetic field is necessary to solve the Einstein field equations to the second order together with the conservation equations.

8 Weakly magnetized FLRW-background

In this section we consider the background space-time as a single barotropic fluid permeated by a weak magnetic field. therefore the energy momentum tensor is formed by two components, the fluid and electromagnetic energy momentum tensor [66, 67]. We allow the presence of a weak magnetic field, this field has the property $B_{(0)}^2 \ll \mu_{(0)}$ and must to be sufficiently random, to satisface $\langle B_i \rangle = 0$ and $\left\langle B_{(0)}^2 \right\rangle = \left\langle B_i^{(0)} B_{(0)}^i \right\rangle \neq 0$ to ensure that symmetries and the evolution of the background remain

unaffected.[9] Again we work under MHD approximation, thus in large scales the plasma is globally neutral, charge density is neglected and the electric field with the current should be zero, thus the only zero order magnetic variable is $B_{(0)}^2$ [67]. Bianchi models are often used to describe the presence of a magnetic field in the universe due to anisotropic properties of this metric. However, as we are dealing with weak magnetic fields, is worth to assume the presence of a magnetic field in a FLRW metric as background.[10] Indeed, the authors in [85] found that, though there is a profound distinction between the Bianchi I equations and the FLRW approximation, at the weak field limit, these differences are reduced dramatically, and therefore the linearised Bianchi equations are the same with the FLRW ones. Under these conditions, we find that to zero order the electromagnetic energy momentum tensor in the background is given by:

$$T_{(em)\,0}^0 = -\frac{1}{8\pi}B_{(0)}^2, \tag{8.1}$$

$$T_{(em)\,i}^0 = T_{(em)\,0}^i = 0, \tag{8.2}$$

$$T_{(em)\,l}^i = \frac{1}{24\pi}B_{(0)}^2\delta_l^i. \tag{8.3}$$

The magnetic anisotropic stress is treated as a first-order perturbation due to stochastic properties of the field, therefore it does not contribute to the above equations.[11] We can see in eqs. (3.5) and (8.1)-(8.3), that fluid and electromagnetic energy-momentum tensor are diagonal tensors, that is, are consistent with the condition of an isotropic and homogeneous background [67]. If we consider the average magnetic density of the background different to zero, the perturbative expansion at $k-$th order of the magnetic density is given by

$$B^2 = B_{(0)}^2 + \sum_{k=1}^{\infty}\frac{1}{k!}B_{(k)}^2, \tag{8.4}$$

where at first order we get a gauge invariant term which describes the magnetic energy density

$$\Delta_{mag}^{(1)} \equiv B_{(1)}^2 + \left(B_{(0)}^2\right)' \mathcal{S}_{(1)}^{||}; \tag{8.5}$$

one can find that average density of the background field decays as $B_{(0)}^2 \sim \frac{1}{a^4(\tau)}$ [86]. At first orden we work with finite conductivity (real MHD), in this case the electric field and the current becomes nonzero, therefore using the eq. (3.7) and assuming the ohmic current is not neglected, we find the Ohm's law

$$J_i^{(1)} = \sigma\left[E_i^{(1)} + \left(\mathcal{V}^{(1)}\times B^{(0)}\right)_i\right]. \tag{8.6}$$

The electromagnetic energy momentum tensor is given by

$$T_{(em)\,0}^0 = -\frac{1}{16\pi}F_{(1)}^2, \tag{8.7}$$

$$T_{(em)0}^i = \frac{1}{4\pi}\left[B_{(0)}^2\vartheta_{(1)}^i - \epsilon^{ikm}E_k^{(1)}B_{(0)}^m + B_{(0)}^2\left(\chi_{\perp(1)}^i\right)'\right], \tag{8.8}$$

$$T_{(em)i}^0 = \frac{1}{4\pi}\left[\epsilon_i^{km}E_k^{(1)}B_{(0)}^m\right], \tag{8.9}$$

$$T_{(em)\,l}^i = \frac{1}{4\pi}\left[\frac{1}{12}F_{(1)}^2\delta_l^i + \Pi_{l(em)}^{i(1)}\right], \tag{8.10}$$

where

$$F_{(1)}^2 = 2\Delta_{(mag)}^{(1)} - 8\Phi^{(1)}B_{(0)}^2 - 2\left(B_{(0)}^2\right)'\mathcal{S}_{(1)}^{||} + \frac{4}{3}\nabla^2\chi^{(1)}B_{(0)}^2 - 8H\mathcal{S}_{(1)}^{||}B_{(0)}^2, \tag{8.11}$$

---

[9]we assume that at zero order the magnetic field has been generated by some random process which is statistically homogeneous so that $B_{(0)}^2$ just depend on the time, $\langle..\rangle$ denotes the expectation value.

[10]In fact, this is what we do every time we study the radiation era. The stress energy tensor of a random radiation field is nothing else but the electromagnetic energy momentum tensor without it's off-diagonal terms.

[11]Again, we omit the brackets because the density evolves exactly like field itself [67, 73].

and $\Pi^{i(1)}_{l(em)} = \frac{1}{3}\left(\Delta^{(1)}_{mag} + E^2\right)\delta^i_l - B_lB^i - E_lE^i$ is the anisotropic stress which appear as a perturbation of the background, this term is important to constrains the total magnetic energy because it's source of gravitational waves [33]. The above equations are written in terms of gauge invariant variables plus terms as $\mathcal{S}_{(1)}$ which are gauge dependent. Now, using the above equations eqs. (5.2),(5.4),(5.3) and (5.5) with the ohm's law eq. (8.6), we arrive to the dynamo equation which give us the evolution of magnetic field to first order

$$\left(B^{(1)}_k\right)' + 2HB^{(1)}_k + \eta\left[\nabla\times\left(\nabla\times B^{(1)} - \left(E^{(1)}\right)' - 2HE^{(1)}\right)\right]_k + \left(\nabla\times\left(B_{(0)}\times\mathcal{V}_{(1)}\right)\right)_k = 0. \quad (8.12)$$

When we suppose a weak magnetic field on the background, in dynamo equation appears a new term called *dynamo term* which can amplify the magnetic field, this term depends of the evolution of $\mathcal{V}_{(1)}$, see eq. (4.32), and also from eq. (4.32), seems likely when matter and velocity perturbation grow the dynamo term amplifies the magnetic field, it is a difference with the first approach where the dynamo term just appears at second order. Using eq. (3.7) we find to second order

$$\mathcal{J}^{(2)}_i - 4J^{(1)}_i\Phi^{(1)} - \varrho^{(1)}v^{(1)}_i + \boldsymbol{S}^1_i(\mathcal{O}^{(2)})$$
$$= 2\sigma\left(\frac{1}{2}\left(\mathcal{V}_{(2)}\times B^{(0)}\right)_i + \left(\mathcal{V}_{(1)}\times B^{(1)}\right)_i - 2E^{(1)}_i\left(\Phi^{(1)} - \frac{1}{2}\Psi^{(1)}\right)\right.$$
$$\left. + \frac{1}{2}\mathcal{E}^{(2)}_i - \left(2\Phi^{(1)} + \Psi^{(1)}\right)\left(\mathcal{V}_{(1)}\times B^{(0)}\right)_i + \boldsymbol{S}^2_i(\mathcal{O}^{(2)})\right). \quad (8.13)$$

In this case we see that 3-current has new terms depending on background magnetic field. Doing the expansion to second order in the electromagnetic energy momentum tensor, one can find the following

$$T^0_{(em)\,0} = -\frac{1}{4\pi}\left(E^2_{(1)} - \frac{1}{2}F^2_{(2)} - \left(B^m_{(0)}E^{(1)}_k\right)\epsilon^{ikm}\vartheta^{(1)}_i + \boldsymbol{S}^8(\mathcal{O}^{(2)})\right), \quad (8.14)$$

$$T^i_{(em)0} = \frac{1}{4\pi}\left[B^2_{(0)}\vartheta^i_{(2)} - \epsilon^{ikm}\mathcal{E}^{(2)}_k B^m_{(0)} - \epsilon^{ikm}E^{(1)}_k B^m_{(1)} + \boldsymbol{S}^i_9(\mathcal{O}^{(2)})\right], \quad (8.15)$$

$$T^0_{(em)i} = \frac{1}{4\pi}\left[\epsilon^{km}_i\mathcal{E}^{(2)}_k B^m_{(0)} + \epsilon^{km}_i E^{(1)}_k B^m_{(1)} + \boldsymbol{S}_{10i}(\mathcal{O}^{(2)})\right], \quad (8.16)$$

$$T^i_{(em)\,l} = \frac{1}{4\pi}\left[\frac{1}{3}\left(\frac{1}{2}F^2_{(2)} - 2B^2_{(0)}\left(\Phi^{(2)} - 2\left(\Phi^{(1)}\right)^2\right) - 4\Delta^{(1)}_{mag}\Phi^{(1)} + E^2_{(1)} + \Delta^{(2)}_{mag}\right.\right.$$
$$\left.\left. - \left(B^m_{(0)}E^{(1)}_k\right)\epsilon^{ikm}\vartheta^{(1)}_i + \boldsymbol{S}^i_{4l}(\mathcal{O}^{(1)})\right)\delta^i_l + \Pi^{i(2)*}_{l(em)} + \boldsymbol{S}^i_{11l}(\mathcal{O}^{(2)})\right], \quad (8.17)$$

where

$$F^2_{(2)} = 2B^2_{(0)}\left(\Phi^{(2)} - 2\left(\Phi^{(1)}\right)^2\right) + 4\Delta^{(1)}_{mag}\Phi^{(1)} + E^2_{(1)} - \Delta^{(2)}_{mag} - 2\epsilon^{ijk}\vartheta^{(1)}_i B^k_{(0)}E^{(1)}_j - 4\left(B^2_{(0)}\right)'\mathcal{S}^{||}_{(1)}\Phi^{(1)}$$
$$- 4\left(H\mathcal{S}^{||}_{(1)} - \frac{1}{6}\nabla^2\chi^{(1)}\right)^2 + 2B^2_{(0)}\left(H\mathcal{S}^{||}_{(2)} - \frac{1}{6}\nabla^2\chi^{(2)} - \mathcal{T}^2(\mathcal{O}^{(2)})\right) - 8\Phi^{(1)}\left(H\mathcal{S}^{||}_{(1)} - \frac{1}{6}\nabla^2\chi^{(1)}\right)$$
$$+ 4\left(\Delta^{(1)}_{mag} - \left(B^2_{(0)}\right)'\mathcal{S}^{||}_{(1)}\right)\left(H\mathcal{S}^{||}_{(1)} - \frac{1}{6}\nabla^2\chi^{(1)}\right) - 2\epsilon^{ijk}\left(\chi^{\perp(1)}_i\right)'B^k_{(0)}E^{(1)}_j + \mathcal{T}^4(\mathcal{O}^{(2)}). \quad (8.18)$$

Using the Maxwell equations together with the Ohm's law to second order eq. (8.13) and following the same methodology for the first order case, we obtain the follow relation for the evolution of magnetic

fields at second order:

$$\left(\mathcal{B}_k^{(2)}\right)' + 2H\left(\mathcal{B}_k^{(2)}\right) + \eta\Bigg[\nabla \times \Big(\frac{1}{a}\Big(\left(\nabla \times \mathcal{B}^{(2)}\right) - 2E_{(1)}\left(2\left(\Psi^{(1)}\right)' - 6\left(\Phi^{(1)}\right)'\right)$$

$$- \left(\mathcal{E}_{(2)}\right)' - 2H\mathcal{E}_{(2)} - 2\left(\nabla \times B_{(1)}\left(2\Psi^{(1)} - 6\Phi^{(1)}\right)\right) - \boldsymbol{S}_{15}\left(\mathcal{O}^{(1)}\right)\Big) - \varrho^{(1)}v^{(1)} + \boldsymbol{S}^1\left(\mathcal{O}^{(2)}\right)\Big)\Bigg]_k$$

$$+ \Bigg(\nabla \times \Big[-4\left(E^{(1)} + \mathcal{V}_{(1)} \times B^{(0)}\right)\Phi^{(1)} - \left(\mathcal{V}_{(2)} \times B^{(0)}\right) + \left(2\Phi^{(1)} + \Psi^{(1)}\right)\left(\mathcal{V}_{(1)} \times B^{(0)}\right)$$

$$- 2\left(\mathcal{V}_{(1)} \times B^{(1)}\right) + 4E^{(1)}\left(\Phi^{(1)} - \frac{1}{2}\Psi^{(1)}\right) - 2\boldsymbol{S}^2\left(\mathcal{O}^{(1)}\right)\Big]\Bigg)_k = -\boldsymbol{S}_k^{17}\left(\mathcal{O}^{(2)}\right). \qquad (8.19)$$

New terms arise because the magnetic density is nonzero on the background, giving origin to terms as $\left(\mathcal{V}_{(2)} \times B^{(0)}\right)$ and couples with geometrical terms $\left(2\Phi^{(1)} + \Psi^{(1)}\right)\left(\mathcal{V}_{(1)} \times B^{(0)}\right)$ which could contribute to the amplification of the field at small scales. In this approach does not appear fractional orders in the variables making a more consistent treatment of the perturbation theory introduced in section 2.

9 Discussion

A problem in modern cosmology is to explain the origin of cosmic magnetic fields. The origin of these fields is still in debate but they must affect the formation of large scale structure and the anisotropies in the cosmic microwave background radiation (CMB) [87–89]. We can see this effect in eq. (6.26) where the evolution of $\Delta_\mu^{(2)}$ depends on the magnetic field. In this paper we show that the perturbed metric plays an important role in the global evolution of magnetic fields [90]. From our analysis, we wrote a dynamo like equation for cosmic magnetic fields to second order in perturbation theory in a gauge invariant form. We get the dynamo equation from two approaches. First, using the FLRW as a background space-time and the magnetic fields as a perturbation. The results are eqs. (5.6) and (7.5) to second order. The second approach a weakly magnetic field was introduced in the background space-time and due to it's statistical properties which allow us to write down the evolution of magnetic field eqs. (8.12) and (8.19) and fluid variables in according to [67]. We observe that essentially, the functional form are the same in the two approaches, the coupling between geometrical perturbations and fields variables appear as sources in the magnetic field evolution giving a new possibility to explain the amplification of primordial cosmic magnetic fields. One important distinction between both approximations is the fractional order in the fields which appears when we consider the magnetic variables as perturbations on the background at difference when the fields are from the beginning on the background (section 8). Although the first alternative is often used in studies of GWs production in the early universe [75, 76], the physical explanation of these fractional orders is sometimes confused, while if we consider an universe permeated with a magnetic density from the background, the perturbative analysis is more straightforward. Further studies as anisotropic (Bianchi I) and inhomogeneous (LTB) models should be addressed to see the implications from the metric behaviour in the evolution of magnetic field and relax the assumption in the weakness of the field.

10 Acknowledgments

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